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# Efficient Frontier and Lower Partial Moment of the First Order

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***Abstract***- In this paper after a quick review on the concept of Efficient Frontier (EF), it is discussed how to derive EF on the basis of Lower Partial Moment of the first order. Then shape of the new family of EFs is investigated. This is a contribution to the literature as no such method is known to exist.

***Keywords***- *Finance, Efficient frontier; Lower Partial Moments; Portfolio Selection; Evolutionary Algorithms*

## I. INTRODUCTION

Investors including large institutions such as mutual funds and pension funds use portfolio management systems to support their asset allocations. In this regard deriving EF on the basis of historical information is an essential initial step to remove inefficient portfolios, otherwise the complexity of decision making increases considerably [1]. A portfolio is efficient if there is no other portfolio with same or higher expected return and lower risk; the collection of portfolios with this property is called efficient set or EF. On the important position of EF in the field of portfolio selection it is good to refer to Ballestero and Romero [2] and Jasemi et al. [3] that recommend maximizing investors expected utility on EF to come to the best choice for investment.

A typical modeling of EF that must be solved for different amounts of $R_d$ is as follows:

$$Min \quad Risk\left(P\left(x_1,\ldots,x_n\right)\right)$$

$$\sum_{i=1}^{n} x_i = 1 \tag{1}$$

$$\sum_{i=1}^{n} x_i \bar{r}_i = R_d \tag{2}$$

$$\sum_{i=1}^{n} y_i = a \tag{3}$$

$$l_i y_i \le x_i \le u_i y_i \quad i = 1,\ldots,n \tag{4}$$

$$y_i = \begin{cases} 1 & x_i > 0 \\ 0 & x_i = 0 \end{cases} \quad i = 1,\ldots,n \tag{5}$$

$$x_i \ge 0 \quad i = 1,\ldots,n \tag{6}$$

Where

$Risk$ : Risk function.

$x_i$ : Share of stock $i$ in the portfolio.

$P(x_1,\ldots,x_n)$: The portfolio whose stocks shares are $x_1,\ldots,x_n$.

$\bar{r}_i$ : Indicator of stock $i$ past returns performance.

$a$ : Desired number of stocks in the portfolio.

$l_i$ and $u_i$ : Lower and upper bound of $x_i$; $i = 1, 2, \ldots, n$ .

In the model, constraints 1, 2 and 6 are mandatory while the others that are also discussed mathematically by Perold [4] are not.

In this study the $Risk$ is Lower Partial Moment (LPM) of the first order, which Fishburn [5] believes to suit a risk-neutral investor, Spreitzer and Reznik [6] has some discussions about and Jasemi et al. [7] proves it to be sensibly coherent.

## II. EF AND ITS NEW RISK MEASURE

### A. LPM

The so called $(\alpha,\tau)$ modeling of LPM, Eq.7, developed by Fishburn (1997);

$$LPM_{\alpha}(\tau;R) = \int_{-\infty}^{\tau} (\tau - R)^{\alpha} dF(R) = E\left\{\left(\max\left[\tau - R, 0\right]\right)^{\alpha}\right\} \tag{7}$$

Where $F(R)$ is the cumulative distribution function, $\tau$ is the target parameter and $\alpha$ that in this study equals "1" determines the weight of deviations.

The LPM family of risk measures is of special importance for application to financial decision making in the way that Bawa [8,9], Harlow and Rao [10], and Unser [11] firmly recommend its application for development of asset pricing models.

### B. Risk of a portfolio

To calculate $LPM_1(\tau,R)$ of $P(x_1,\ldots,x_n)$ two approaches can be devised as Eq.s 8 and 9 while as it is obvious the former is stock-driven and the latter is portfolio-driven.

$$\sum_{i=1}^{n} x_i \times \left[\int_{-\infty}^{\tau} (\tau - r_i) f_i(r_i) dr_i\right], \tag{8}$$

$$\int_{-\infty}^{\tau}\left(\tau-\sum_{i=1}^{n}x_i r_i\right) f_{RP(x_1,\ldots,x_n)}\left(\sum_{i=1}^{n}x_i r_i\right) d\left(\sum_{i=1}^{n}x_i r_i\right) \qquad (9)$$

where $RP(x_1,\ldots,x_n)$ is return of $P(x_1,\ldots,x_n)$.

Since for the first and the second approach there are $n$ and infinite distribution functions to be estimated respectively, application of Eq.8 in the model is much simpler but there is a main problem with it. The problem arises when return of an asset is smaller (bigger) than $\tau$ but the weighted average of the portfolio return is bigger (smaller) than $\tau$. To exemplify the problem consider two following independent variables of $r_1$ and $r_2$.

$$\Pr_1(r_1)=\begin{cases}50\% & r_1=-10\%\\ 50\% & r_1=30\%\end{cases}, \qquad \Pr_2(r_2)=\begin{cases}30\% & r_2=0\\ 70\% & r_2=20\%\end{cases};$$

so the probability function of the portfolio in which $x_1 = 20\%$ and $x_2 = 80\%$, would be as follows:

$$\Pr_p(RP)=\begin{cases}15\% & RP=-2\%\\ 35\% & RP=14\%\\ 15\% & RP=6\%\\ 35\% & RP=22\%\end{cases};$$

and $LPM_1(15\%, RP(20\%,80\%)) = 4.25\%$ while Eq.8 calculates it as 6.1%.

To approximate $LPM_1(\tau, RP(x_1,\ldots,x_n))$ by Eq.9, first of all $f_{RP(x_1,\ldots,x_n)}$ must be estimated. Here it is done by applying the concept of histograms. For our problem, on the basis of Bowker and Lieberman [12] two methods seem more appropriate for drawing the intended histograms. In one, the intervals length of the histogram is specified in the way that it displays a plane image that ascends evenly before the maximum point and after reaching it, descends evenly. Obviously between two different even histograms, the one with more intervals is preferred. In the other method that has been proved to be more appropriate [13] and is going to be applied in our EF model, the intervals are too short to encompass more than one distinct data. Fig.1 depicts a typical one where $r_i$ denotes the $i$ th smallest return of the asset, $f_i$ determines frequency of $r_i$ and $N$ is the number of different returns of the asset.

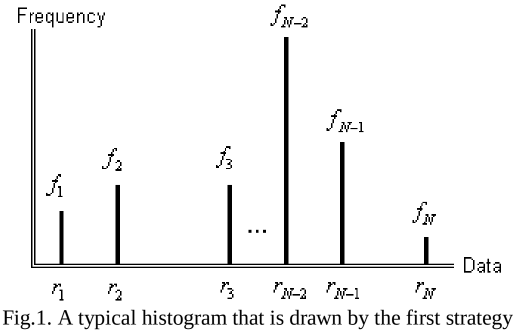


Fig.1. A typical histogram that is drawn by the first strategy

To calculate $LPM_1(\alpha,\tau)$ by the second method (Fig.1), it should be converted to its equivalent discrete version, Eq.10.

$$LPM_1(\tau, R) \approx \sum_{-100}^{\tau}(\tau - R)\Pr(R). \qquad (10)$$

Now if $r_k < \tau < r_{k+1}$, $LPM_1(\tau, R)$ is calculated by Eq. (11).

$$\sum_{-100}^{\tau}(\tau-R)P(R)=\sum_{i=1}^{k}(\tau-r_i)\frac{f_i}{\sum_{i=1}^{n}f_i}=\frac{\sum_{i=1}^{k}(\tau-r_i)f_i}{\sum_{i=1}^{n}f_i}. \qquad (11)$$

If it is assumed that the time horizon is of length $T$, return of $P(x_1,\ldots,x_n)$ on the $t^{\text{th}}$ time unit is calculated by Eq.12.

$$RP_t(x_1,\ldots,x_n)=x_1.ar_{1t}+x_2.ar_{2t}+\ldots+x_n.ar_{nt}=\sum_{i=1}^{n}x_i.ar_{it} \qquad t=1,2,\ldots,T, \qquad (12)$$

where $ar_{it}$ is return of asset $i$ on the $t^{\text{th}}$ time unit. Then the final EF model, however in its simplest form without the cardinality and bounding constraints, is as follows:

$$Min \ \frac{\sum_{i=1}^{k}\left(\tau - r_{pi}(x_1,\ldots,x_n)\right)f_i}{\sum_{i=1}^{n}f_i}$$

$$\sum_{i=1}^{n}x_i = 1$$

$$\sum_{i=1}^{n}x_i\bar{r}_i = r_d$$

$$x_i \ge 0 \quad i=1,2,\ldots,n$$

where $r_{pi}(x_1,\ldots,x_n)$ is the $i^{\text{th}}$ smallest return of $P(x_1,\ldots,x_n)$.

In the above formulation the parameters $k$, $r_{pi}$ and $f_i$ are functions of $(x_1, \ldots, x_n)$; i.e. for each set of amounts for the variables there will be a new objective function that makes us use the evolutionary approach of Genetic Algorithm to solve the model.

### *C. Running the new model*

As a matter of fact, $\bar{r}_i$ in Eq.2 is decided to be "Arithmetic average of past returns that are greater than $\tau$" to meet the two important following criteria:

- All data should be equal from the perspective of the times being applied by the model.
- Both characteristics of Mean and Volatility being considered.

To survey the general shapes of the new EFs, three following sections are devised. In the first and second ones two and more than two (including six and eighteen) assets respectively, are considered while in the third section there are eighteen assets with cardinality and bounding constraints.

*C.1. Two asset EF*

In this part three two-asset combinations from NYSE with different correlation ratios between their daily returns are considered. Table 1 and Fig.2 correspond to Dell and Hp in 2007 with correlation ratio of +0.51 and $\tau = 2\%$.

Table 1: EF with 20 portfolios

| Point | HP | Dell | LPM(%) | Return(%) |
|---|---|---|---|---|
| **1** | 0.97 | 0.03 | 1.971 | 2.816 |
| **2** | 0.97 | 0.03 | 1.972 | 2.82 |
| **3** | 0.93 | 0.07 | 1.972 | 2.825 |
| **...** | ... | ... | ... | ... |
| **18** | 0.14 | 0.86 | 2.057 | 2.893 |
| **19** | 0.09 | 0.92 | 2.07 | 2.897 |
| **20** | 0.03 | 0.97 | 2.084 | 2.902 |

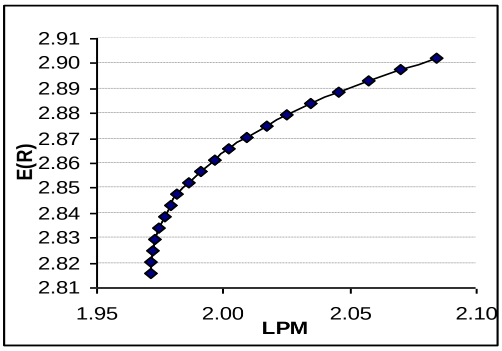


Fig.2: The EF of Table 1

Table 2 and Fig.3 correspond to Boeing and Microsoft in 1999 with correlation ratio of +0.054 and $\tau = 0$.

Table 2: EF with 19 portfolios

| Point | Boeing | Microsoft | LPM(%) | Return(%) |
|---|---|---|---|---|
| **1** | 0.58 | 0.42 | 2.067 | 1.852 |
| **2** | 0.54 | 0.45 | 2.068 | 1.869 |
| **3** | 0.51 | 0.49 | 2.069 | 1.887 |
| **...** | ... | ... | ... | ... |
| **17** | 0.07 | 0.93 | 2.273 | 2.126 |
| **18** | 0.04 | 0.96 | 2.299 | 2.143 |
| **19** | 0 | 1 | 2.325 | 2.16 |

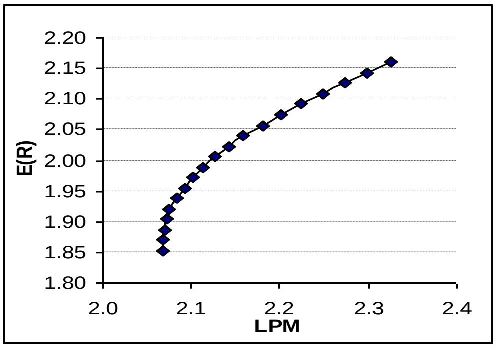


Fig.3: The EF of Table 2

The daily returns of General Electric (GE) and Honda during 2007 have the correlation ratio of +0.456 but to see the shape of EF when correlation ratio is not positive, negative of Honda stock returns is considered and so the new correlation ratio becomes -0.456. For this case Table 3 and Fig.4 show the EF for $\tau = 1\%$.

Table 3: EF with 20 portfolios

| Point | GE | -Honda | Return(%) | LPM(%) |
|---|---|---|---|---|
| **1** | 0.99 | 0.01 | 1.815 | 1.121 |
| **2** | 0.94 | 0.06 | 1.824 | 1.097 |
| **3** | 0.88 | 0.12 | 1.832 | 1.074 |
| **...** | ... | ... | ... | ... |
| **18** | 0.12 | 0.88 | 1.957 | 1.107 |
| **19** | 0.07 | 0.94 | 1.965 | 1.136 |
| **20** | 0.01 | 0.99 | 1.974 | 1.169 |

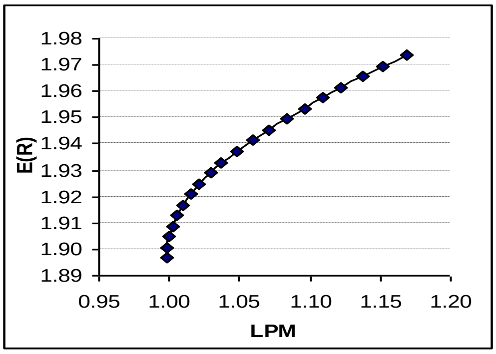


Fig.4: The EF of Table 3

*C.2. More than two asset EF*

Table 4 and Fig.5 correspond to six stocks of HP, Dell, Boeing, Microsoft, GE and Honda in NYSE during 2007 and $\tau = 2\%$. Each point in the table corresponds to an efficient portfolio while the figure represents the associated EF.

Table 4: EF with 20 portfolios

| Point | HP | Dell | Boeing | Microsoft | GE | Honda | LPM(%) | Return(%) |
|---|---|---|---|---|---|---|---|---|
| **1** | 0.019 | 0.009 | 0.312 | 0.619 | 0.032 | 0.01 | 1.976 | 2.969 |
| **2** | 0.015 | 0.007 | 0.287 | 0.657 | 0.025 | 0.008 | 1.977 | 2.986 |
| **3** | 0.011 | 0.004 | 0.285 | 0.674 | 0.021 | 0.005 | 1.977 | 2.995 |
| **…** | … | … | … | … | … | … | … | … |
| **18** | 0 | 0 | 0.034 | 0.966 | 0 | 0 | 1.995 | 3.135 |
| **19** | 0 | 0 | 0.017 | 0.983 | 0 | 0 | 1.996 | 3.144 |
| **20** | 0 | 0 | 0 | 1 | 0 | 0 | 1.998 | 3.153 |

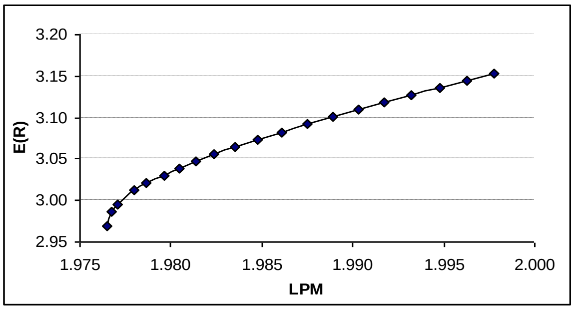

Fig.5: The EF of Table 4

About the case with eighteen assets moreover to the above six stocks, three stocks of GM, IBM and Nike from NYSE are also considered. The nine stocks are analyzed during 2006 and 2007 in the way that each stock in a year is considered as an independent asset. The resulted EFs for $\tau = 0$ and $\tau = 1\%$ are shown in Fig.6 while risks and returns are presented in percentage.

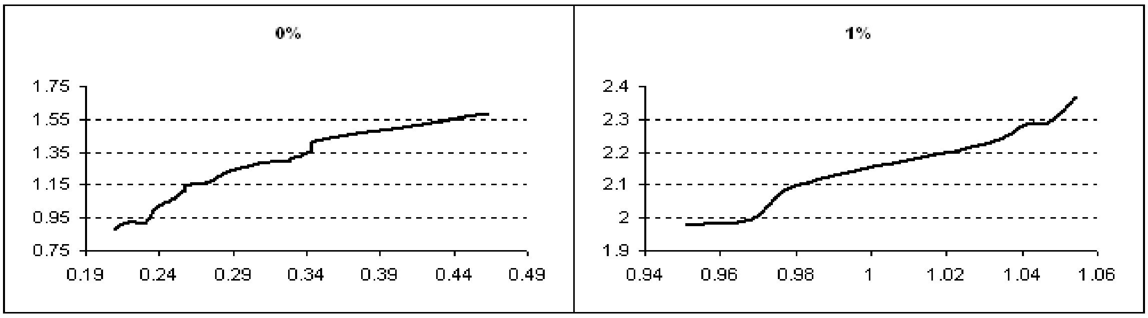

Fig.6: Two EFs

*C.3. The complete EF model*

In this section it is investigated that what would happen to EF when the number of assets in each portfolio is restricted and the amount of investment in each stock has lower and upper bounds. Here the eighteen assets of previous part are used again while the lower and upper bounds are dependent to $a$ as is formulated by Eq.(13).

$$l_i = \frac{1}{2a}\,;\; u_i = 1 - \frac{a-1}{2a} \quad i = 1, 2, ..., n\,. \qquad (13)$$

The EFs of this section are derived with three amounts of $a$ as 6, 12 and 18. Fig.7 shows the results for $\tau = 0$ and $\tau = 1\%$.

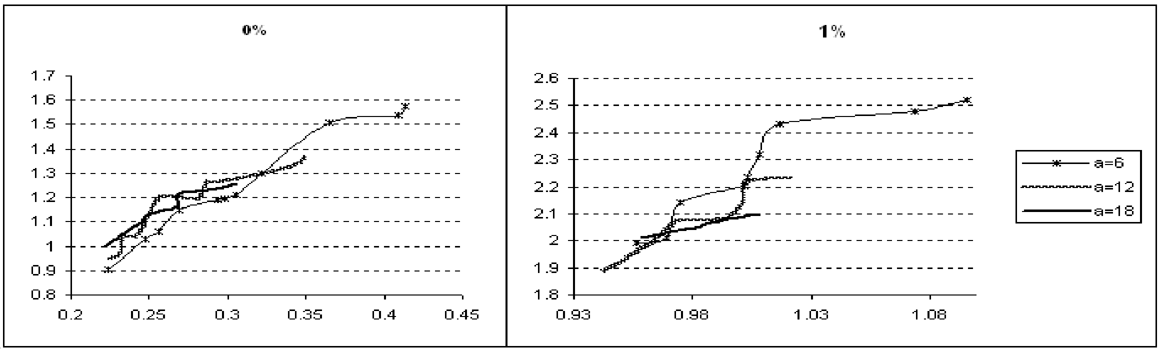

Fig.7: Six EFs of two amounts of $\tau$ and three amounts of $a$.

## III. CONCLUSION

The concept of EF was the main focus of this paper. The difference between this study and the others of the field can be summarized in the two following items.

- Considering the risk measure of LPM of the first order for deriving EF.
- Presenting a practical approach to derive EF on the basis of the LPM while the approach is not restricted by factors like stochastic characteristics of the stocks returns or number of stocks that compose the portfolio.

The proposed mechanism is applied to derive EF for diverse amounts of $\tau$ s and constraints combinations to give a general understanding of the new generation of EF. The performance of the proposed mechanism is firmly approved by the sensible results from the perspectives of shape and technical acceptance. The general shape of the EFs is a concave ascending parabola; i.e. the riskier the portfolio, more return can be expected.

At last it is highly recommended to replace Variance in any financial model with LPM of the first order, then a comprehensive comparison between them being conducted and the results being analyzed. Surely in this path there is a potential for improvement of financial models.

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