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# Observer Based Control of a Class of Nonlinear Fractional Order Systems Using LMI

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***Abstract***- Design of an observer based controller for a class of fractional order systems has been done. Fractional order mathematics is used to express the system and the proposed observer. Fractional order Lyapunov theorem is used to derive the closed-loop asymptotic stability. The gains of the observer and observer based controller are derived systematically using the linear matrix inequality approach. Finally, the simulation results demonstrate validity and effectiveness of the proposed observer based controller.

**Keywords-** *Fractional order calculus, Fractional order observer, Linear matrix inequality, Nonlinear Systems, Observer based Controller.*

## I. INTRODUCTION

Fractional order calculus, an old mathematical topic from the 17th century, has recently attracted a rapid growth in the number of applications where fractional calculus has been used [1]-[5].

The design of state estimators is one of the essential points in control theory and the observer-based control is usually applied when we do not have access to all the states of a system [6]. There are a few researches on the fractional order observer based controls of the fractional order system, Both in linear case [6], [7] and nonlinear ones [7]. Some papers introduced synchronization of chaotic systems using observer [8-11]. Almost all of the previous work has ignored nonlinearity or removed it by use of the designed controller.

The major difficulties in the design of practical observers for dynamical systems are their nonlinear dynamics which may results in failure of practical use of previous methods. This means that designing fractional observer or observer based controller for nonlinear fractional order systems are still an open problem.

To the best of our knowledge, [7] is the lone reference that introduced designing observer based controller for nonlinear affine fractional order systems by considering nonlinearity that used Gronwall Bellman lemma in the proof procedure. This reference has considers some assumptions on nonlinear function and state's initial condition besides a complex stability proof that restrict its usage.

Besides, for extending the application of fractional calculus in nonlinear systems, [12] propose the fractional Lyapunov direct method with a view to enrich the knowledge of both system theory and fractional calculus. The main interest of Lyapunov's approach is to define Linear Matrix Inequalities (LMIs) conditions. But it is also well known that Lyapunov's technique is the fundamental tool to analyze the stability of nonlinear systems [13].

In this paper we consider Lipschitz nonlinear fractional order systems. Our objective is to find an observer based controller that stabilizes the state estimation error. An LMI based observer gain for this class of nonlinear systems has derived using fractional direct Lyapunov theorem.

This paper is organized as follows: Section II provides preliminary definitions. In section III, the nonlinear fractional order observer is given and the design procedure for observer based controller is discussed. Numerical example is provided in section IV and finally, the conclusion remarks are given.

## II. PRELIMINARY DEFINITIONS

In this section we recall the main definitions and results concerning fractional calculus.

***Definition 1:*** [2], [14] One of the basic functions of the fractional calculus is Euler's Gamma function which is defined by

$$\Gamma(z)=\int_0^{\infty} e^{-t}t^{z-1}dt \tag{1}$$

which converges in the right half of the complex plane.

***Definition 2:*** [2], [14] The $q$ th-order Riemann-Liouville fractional derivative of function $f(t)$ with respect to t and the initial value a is given by

$$_aD_t^q f(t)=\frac{1}{\Gamma(m-q)}\left(\frac{d}{dx}\right)^m\int_a^t\frac{f(\tau)d\tau}{(t-\tau)^{1+q-m}} \tag{2}$$

where $m$ is the first integer larger than $q$, i.e. $m-1\le q<m$ and $\Gamma$ is the Gamma function.

***Remark 1:*** The $q$ th-order fractional derivative of function

$f(x(t)) = x(t)^2$ with respect to $t$ is given by [15],

$$ {}_0D_t^q f(x(t)) = x(t)\,{}_0D_t^q x(t) + p_x \tag{3}$$

While

$$ p_x = \sum_{k=1}^{\infty} \frac{\Gamma(1+q)}{\Gamma(1+k)\Gamma(1-k+q)} ({}_0D_t^k x)({}_0D_t^{q-k} x) \tag{4}$$

We can consider the following boundedness condition:

$$ \|p_x\| \le \beta \|x\|^2 \tag{5}$$

***Lemma 1 (Schur complement):*** [16] The LMI:

$$ \begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} < 0 \tag{6}$$

Where $Q(x) = Q^T(x),\ R(x) = R^T(x)$, and $S(x)$ affinely depend on $x$, is equivalent to:

$$ \begin{cases} R(x) < 0 \\ Q(x) - S(x)R^{-1}(x)S^T(x) < 0 \end{cases} \tag{7}$$

***Lemma 2:*** [17] Let x, y be real vectors of the same dimension. Then, for any scalar $\varepsilon > 0$, the following inequality holds:

$$ x^T y \le \varepsilon x^T x + \varepsilon^{-1} y^T y \tag{8}$$

***Lemma 3 (Fractional Lyapunov direct method):*** [12], [18] Let $x = 0$ be an equilibrium point for the non-autonomous fractional order system ${}_0D_t^q x(t) = f(t,x)$. Assume that there exists a Lyapunov function $V(t, x(t))$ and class-k functions $\alpha_i (i = 1,2,3)$ satisfying

$$ \alpha_1(\|x\|) \le V(t, x(t)) \le \alpha_2(\|x\|) \tag{9}$$

and

$$ {}_0D_t^{\beta} V(t, x(t)) \le -\alpha_3(\|x\|) \tag{10}$$

where $\beta \in (0,1)$ Then we have $\lim_{t\to\infty} x(t) = 0$.

## III. Observer Based Control for Lipschitz Fractional Order Nonlinear Systems

Consider a nonlinear fractional order system of the form:

$$ \begin{aligned} D^q x &= Ax + Bu + \phi(x,u) \\ y &= Cx \end{aligned} \tag{11}$$

where $x \in \Re^n$, $u \in \Re^q$, and $y \in \Re^m$ are the state, input, and output, respectively, $C \in \Re^{m\times n}$ is constant matrix and $\phi : [\Re^n \quad \Re^q] \to \Re^n$ is nonlinear function that $\phi(0,u) = 0$ and this function is Lipschitz in $x$ with Lipschitz constants $\gamma$, i.e.:

$$ \|\phi(x_1,u) - \phi(x_2,u)\| < \gamma \|x_1 - x_2\| \tag{12}$$

A nonlinear fractional order observer is introduced as:

$$ \begin{aligned} {}_0D_t^q \hat{x} &= A\hat{x} + Bu + \phi(\hat{x},u) + L(y - C\hat{x}) \\ \hat{y} &= C\hat{x} \end{aligned} \tag{13}$$

where $\hat{x}$ is the state estimation and $L$ is the proportional observer gain Then, the observer error dynamic equation is obtained as:

$$ {}_0D_t^q \tilde{x} = (A - LC)\tilde{x} + \phi(x,u) - \phi(\hat{x},u) \tag{14}$$

$\tilde{x} = x - \hat{x}$ is the state estimation error.

In the continue, we study both observation and stabilization of (11) by choosing $u = K\hat{x}$ in which $K$ is the state feedback gain. The following theorem provides sufficient conditions for the stability of the proposed nonlinear observer based fractional order controller.

***Theorem:*** The observer based control $u = K\hat{x}$ has a stable observation and stabilization for the nonlinear system (11) if there exist positive real number $\varepsilon_1$ and matrix $K \in \Re^{1\times n}$ while the proportional observer gain $L \in \Re^{n\times m}$ is the solution of the following constrained LMI:

$$ \begin{bmatrix} (A_1+\theta) + (A_1+\theta)^{\mathrm{T}} & -BK \\ -K^T B^T & (A_2+\theta) + (A_2+\theta)^{\mathrm{T}} \end{bmatrix} < 0 \tag{15}$$

while $A_1 = A + BK$, $A_2 = A - LC$, $\theta = \varepsilon_1^{-1}\gamma + \varepsilon_1 + \beta$ and $\beta$ is a positive constant scalar given in (5).

***Proof:*** Consider the following Lyapunov function candidate:

$$ V = X^T X \tag{16}$$

where $X = [x \quad \tilde{x}]^T$ and we want to investigate stabilization of that. By stabilizing $X$, both $x$ and $\tilde{x}$ will be stabilize and this means robust observation besides stabilization of (11).

Taking the derivative of (16) and using (3), (11) and (14), results in:

$$ \begin{aligned} {}_0D_t^q V &= X^T\,{}_0D_t^q X + p_X \\ &= X^T \begin{bmatrix} Ax + Bu + \phi(x,u) \\ (A - LC)\tilde{x} + \phi(x,u) - \phi(\hat{x},u) \end{bmatrix} + p_X \end{aligned} \tag{17}$$

Using $u = K\hat{x}$, (17) will simplify as:

$$ \begin{aligned} {}_0D_t^q V = X^T ( &\begin{bmatrix} A + BK & -BK \\ 0 & A - LC \end{bmatrix} X + \\ &+ \begin{bmatrix} \phi(x,u) \\ \phi(x,u) - \phi(\hat{x},u) \end{bmatrix}) + p_X \end{aligned} \tag{18}$$

Applying Lemma 2 on the second term, with $\varepsilon_1$ result in:

$$ {}_0D_t^qV \le X^T\begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix}X + \varepsilon_1 X^T X $$
$$ + \varepsilon_1^{-1}\begin{bmatrix} \phi(x,u) \\ \phi(x,u)-\phi(\hat{x},u) \end{bmatrix}^T \begin{bmatrix} \phi(x,u) \\ \phi(x,u)-\phi(\hat{x},u) \end{bmatrix} + p_X \tag{19} $$

Considering (12), the inequality (19) can be rewritten as bellow:

$$ {}_0D_t^qV \le X^T\begin{bmatrix} A+BK+\varepsilon_1^{-1}\gamma+\varepsilon_1 & -BK \\ 0 & A-LC+\varepsilon_1^{-1}\gamma+\varepsilon_1 \end{bmatrix}X \tag{20} $$
$$ + p_X $$

Then using (5) in (20) follows that:

$$ {}_0D_t^qV \le X^T\tilde{A}_1X \tag{21} $$

in which

$$ \tilde{A}_1 \overset{\Delta}{=} \begin{bmatrix} A+BK+\theta & -BK \\ 0 & A-LC+\theta \end{bmatrix} \tag{22} $$

Since $\theta = \varepsilon_1^{-1}\gamma + \varepsilon_1 + \beta$.

Using fractional direct Lyapunov method, the sufficient conditions for asymptotically stability of $X$ is choosing $K, L$ and $\varepsilon_1$ that causes $\tilde{A}_1 < 0$.

Matrix $\tilde{A}_1$ is not a symmetric matrix thus it cannot be converted to LMI by using Lemma 1. In the continue we overcome this problem by replacing $\tilde{A}_1$ with $\tilde{A}_2$ while $X^T\tilde{A}_1X = X^T\tilde{A}_2X$. $\tilde{A}_2$ is defined as bellow:

$$ \tilde{A}_2 \overset{\Delta}{=} \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_2^T & \alpha_4 \end{bmatrix} \tag{23} $$

in which

$$ \alpha_1 = \frac{(A+BK+\theta)+(A+BK+\theta)^{\mathrm{T}}}{2} $$
$$ \alpha_2 = \frac{-BK}{2} \tag{24} $$
$$ \alpha_4 = \frac{(A-LC+\theta)+(A-LC+\theta)^{\mathrm{T}}}{2} $$

To proof the equality of $X^T\tilde{A}_1X = X^T\tilde{A}_2X$, appendix I can be used.

The new condition for asymptotically stability of $X$ is choosing $K, L$ and $\varepsilon_1$ that causes $\tilde{A}_2 < 0$ which yields LMI (15).

## IV. NUMERICAL EXAMPLE

Ninteger is a toolbox for Matlab intended to help developing fractional order controllers and assesses their performance [19]. In this part we introduce a numerical example and use the Matlab/Simulink environment to investigate the proposed observer based controller.

Consider the following unstable nonlinear fractional order system:

$$ D^q x = \begin{bmatrix} -2 & 0.3 & 1 \\ 0 & -2 & 3 \\ 2 & 0 & 0.7 \end{bmatrix} x + \begin{bmatrix} 0.5\sin(x_2) \\ 0.25\sin(2x_3) \\ 0.5e^{-|x_2|} \end{bmatrix} + \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix} u \tag{25} $$
$$ y = [0 \quad 1 \quad 4]x $$

While $x = [x_1, x_2, x_3]^T$ and $q = 0.8$. The design parameters are chosen as $\gamma = 0.7$, $\beta = 0.2$.

Observer (13) with $L = [0.87 \ 109.77 \ 438.43]^T$ and observer based controller $u = K\hat{x}$ with $K = [-0.045 \ -2.16 \ -8.63]$ is analytically stable by theorem 1 while $\varepsilon_1 = 0.69$.

The simulation results for system (25) are shown in Fig. 1, 2 and 3 since the observer is activated at $t = 2s$ and observer based controller is triggered at $t = 3s$.

Fig 1 shows the state estimates in the proposed method since Fig 2 shows the errors of state estimations.

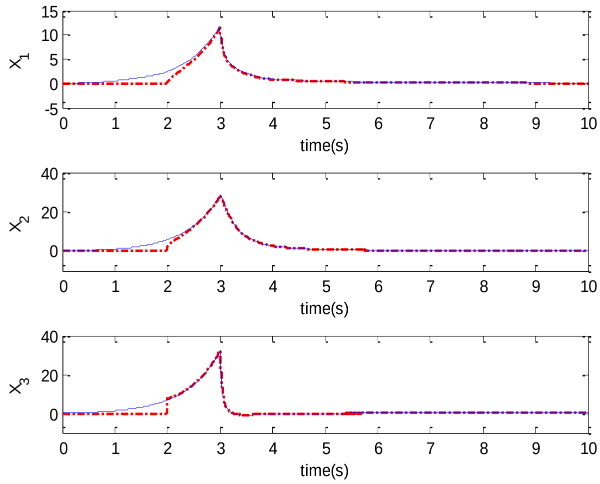


FIG 1. ACTUAL STATES (LINE), STATE ESTIMATIONS (DASHED).

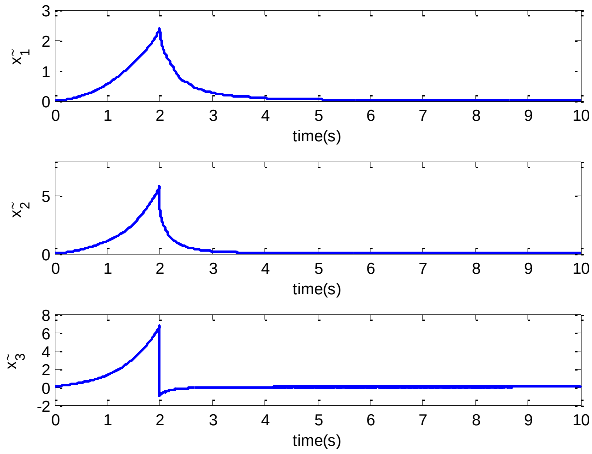


FIG 2. ERRORS OF STATE ESTIMATIONS USING OBSERVER.

As is shown, although the system (25) is unstable, the gain obtained from the proposed observer design causes the estimator to accurately track the system states and the proposed controller can stabilize this system with a small settling time.

Outputs of the system and its observer are shown in fig 3.

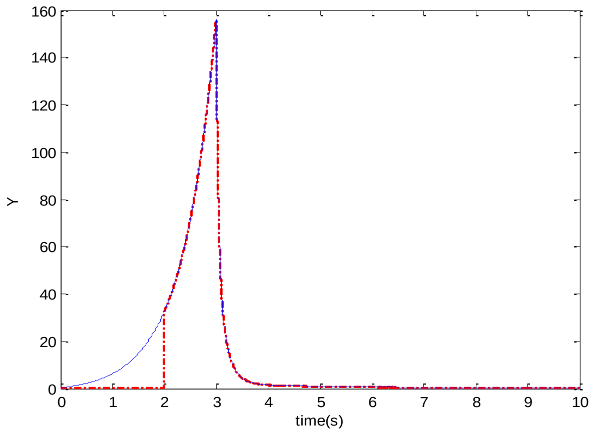


FIG 3. OUTPUT SIGNAL OF SYSTEM (LINE) AND OBSERVER'S OUTPUT (DASHED).

Fig 3 illustrates the efficiency of the observer for $t > 2s$ and the proposed observer based controller besides the observer for $t > 3s$.

## CONCLUSION

We proposed to design a fractional order observer based controller for a class of nonlinear fractional order systems using LMI and fractional order direct Lyapunov theorem.

The proof procedure is explained in detail. Under our scheme, a simple linear controller is used for stabilizing Lipschitz nonlinear systems. Furthermore, the performance of the design, both for observation and control, is satisfactory with acceptable settling that shown in simulation.

## APPENDIX I

For any $x$ and $Q$ we have $x^T Q x \in R$ and $(x^T Qx)^T = x^T Q^T x$ so

$$(x^T Qx)^T = x^T Q^T x = x^T Qx \tag{26}$$

This implies that:

$$x^T \left(\frac{Q-Q^T}{2}\right)x = 0 \tag{27}$$

On the other hand

$$x^T Qx = x^T \left(\frac{Q+Q^T}{2} + \frac{Q-Q^T}{2}\right)x \tag{28}$$

Using (27) will simplify (28) as:

$$x^T Qx = x^T \left(\frac{Q+Q^T}{2}\right)x \tag{29}$$

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